

DEFORMATIONS OF M-THEORY KILLING SUPERALGEBRAS

JOSÉ FIGUEROA-O'FARRILL

ABSTRACT. We classify the Lie superalgebra deformations of the Killing superalgebras of some M-theory backgrounds. We show that the Killing superalgebras of the Minkowski, Freund–Rubin and M5-brane backgrounds are rigid, whereas the ones for the M-wave, the Kaluza–Klein monopole and the M2-brane admit deformations, which we give explicitly.

CONTENTS

1. Introduction	1
2. Lie superalgebra deformations and cohomology	3
2.1. Deformations	3
2.2. Cohomology	4
2.3. Hochschild–Serre factorisation theorem	5
3. Rigidity of the Poincaré superalgebra	6
3.1. Rigidity of the Freund–Rubin superalgebras	9
4. Superalgebra deformations of brane backgrounds	9
4.1. A deformation of the M2-brane Killing superalgebra	9
4.2. Rigidity of the M5-brane Killing superalgebra	13
5. Superalgebra deformations of purely gravitational backgrounds	14
5.1. A deformation of the M-wave Killing superalgebra	14
5.2. Deformations of the Kaluza–Klein monopole Killing superalgebra	16
6. Conclusion	19
Acknowledgments	20
References	20

1. INTRODUCTION

The main article of faith justifying much of the present research on supergravity is that supergravity may teach us something about string theory. In particular, it is assumed that supergravity backgrounds may be corrected to yield exact string backgrounds, something which could perhaps be proved, at least in special cases, using techniques from the study of partial differential equations such as the Banach or Nash–Moser implicit function theorems. We will not question this assumption in this paper. Instead we would like to explore how invariants of a supergravity background can change as the background itself gets deformed to incorporate the quantum corrections. We will focus on one such invariant: the *Killing superalgebra* of the background [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], a Lie superalgebra so called because it is constructed out of Killing vectors and Killing spinors.

EMPG-07-11.

For supergravities which arise as limits of M- or string theories, it is a natural question to ask what happens to the Killing superalgebra under stringy (i.e., α') or M-theoretic quantum corrections. There seems to be some evidence [11, 12, 13, 14, 15, 16, 17, 18] supporting the persistence of the notion of Killing superalgebra under this procedure; although to be fair the study of quantum corrections is still very much in its infancy and there is not enough data to argue this point convincingly. Let us however assume that the notion persists in some way. Then surely one should find the Killing superalgebra of a quantum-corrected background among the *deformations* (in the sense of Gerstenhaber [19]) of the Killing superalgebra of the classical background or, allowing for symmetry breaking, of a suitable subsuperalgebra. It remains to decide what algebraic structure one should deform.

A Lie superalgebra can be viewed in many equivalent ways. It is standard to view it as a vector superspace with a skewsymmetric bracket obeying the Jacobi identity, but by going to the universal enveloping algebra we can also view it as an associative algebra or more generally as a cocommutative Hopf algebra. Conversely every cocommutative Hopf algebra generated by its primitive elements is the universal enveloping algebra of a Lie superalgebra. Dually, we may also view a Lie superalgebra structure on a vector space V as a differential graded superalgebra structure on $\Lambda^\bullet V^*$, whose differential has degree 1. Conversely every such differential is dual to a Lie superalgebra structure on V . The question is then how to deform a Lie superalgebra: as a Lie superalgebra? as a Hopf algebra? or as a differential graded superalgebra? In the first case we remain in the world of Lie superalgebras, whereas the other two cases would bring us to the worlds of quantum supergroups and L_∞ superalgebras, respectively. From our present position of ignorance, the safest assumption and, in any case, the one we will explore in this paper, is to remain within the category of Lie superalgebras.

Therefore in this paper we will classify the possible *Lie superalgebra deformations* of the Killing superalgebras of some M-theory backgrounds: all maximally supersymmetric backgrounds except for the Kowalski-Glikman wave, and the elementary half-BPS backgrounds: M2- and M5-branes, as well as the M-wave and the Kaluza–Klein monopole. The calculations employ established homological techniques which we will briefly review below.

These calculations may also be of use in classical supergravity. Indeed, deformation is an inverse process to that of contraction; that is, the deformations of a Lie superalgebra \mathfrak{g} consist of all the Lie superalgebras which contract to \mathfrak{g} analytically. We know that under certain geometric limits, such as the plane-wave limit [20, 21], the Killing superalgebra of a background gets contracted [22, 23, 24, 25]. Hence classifying the possible deformations of the Killing superalgebra of a background gives us hints about the existence of other nearby backgrounds of which the background in question can be a geometric limit. Of course, reconstructing the background from its Killing superalgebra is only ever possible if the dimension of the superalgebra is large enough to constrain the geometry sufficiently. Research is in progress [26] to investigate the existence of classical M-theory backgrounds whose Killing superalgebras are the deformations found in this paper.

This paper is organised as follows. In Section 2 we will discuss the basics of Lie superalgebra cohomology and the basic technique to compute the possible deformations, based on the Hochschild–Serre spectral sequence. In Section 3 we prove that the Killing superalgebra of the Minkowski background is rigid, in contrast with the four-dimensional situation. Appealing to general results, we deduce in Section 3.1, that the Freund–Rubin superalgebras too are rigid. In Section 4 we explore the Lie superalgebra deformations of the Killing superalgebras for the elementary M2- and M5-brane. We find that whereas the Killing superalgebra of the M5-brane is rigid, that of the M2-brane admits an integrable one-parameter deformation suggesting

that the worldvolume of the membrane deforms to AdS_3 . In Section 5 we do the same for the Killing superalgebras of the elementary half-BPS purely gravitational backgrounds: the M-wave and the Kaluza–Klein monopole, and find that whereas the M-wave superalgebra admits an integrable one-parameter deformation, the Kaluza–Klein monopole superalgebra admits two such families: one is reminiscent of a nongeometric background, whereas the other suggests that the Minkowski factor deforms to AdS_7 . Finally in Section 6 we offer some concluding remarks.

2. LIE SUPERALGEBRA DEFORMATIONS AND COHOMOLOGY

2.1. Deformations. Recall that a Lie superalgebra is a vector superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, together with an even bilinear map $[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ which is *skewsymmetric*

$$[X, Y] = -(-1)^{XY} [Y, X] , \quad (1)$$

and satisfies the *Jacobi identity*

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{XY} [Y, [X, Z]] , \quad (2)$$

for homogeneous $X, Y, Z \in \mathfrak{g}$ and where in the expression for signs we denote the grading of a homogeneous $X \in \mathfrak{g}$ also by X .

By a **Lie superalgebra deformation** of \mathfrak{g} , we mean a one-parameter family of Lie superalgebra structures $[-, -]_t$ on \mathfrak{g} depending analytically on t and agreeing at $t = 0$ with the original Lie superalgebra structure $[-, -]$. Expanding the bracket $[-, -]_t$ in a power series in t we find

$$[X, Y]_t = [X, Y] + t\Phi_1(X, Y) + t^2\Phi_2(X, Y) + \cdots = \sum_{n \geq 0} t^n \Phi_n(X, Y) , \quad (3)$$

with $\Phi_0(X, Y) = [X, Y]$. The skewsymmetry condition (1) says that

$$\Phi_k(X, Y) = -(-1)^{XY} \Phi_k(Y, X) \quad (4)$$

for all k , whereas the Jacobi identity gives rise to an infinite number of equations, one for each power of t :

$$\sum_{\ell+m=n} (\Phi_\ell(X, \Phi_m(Y, Z)) - \Phi_\ell(\Phi_m(X, Y), Z) - (-1)^{XY} \Phi_\ell(Y, \Phi_m(X, Z))) = 0 , \quad (5)$$

for all $n \geq 0$. The first equation, for $n = 0$, is the Jacobi identity for $\Phi_0 = [-, -]$ and for $n > 0$ we obtain equations for the higher Φ_k . In particular, the equation

$$\begin{aligned} \Phi_1(X, [Y, Z]) - \Phi_1([X, Y], Z) - (-1)^{XY} \Phi_1(Y, [X, Z]) \\ + [X, \Phi_1(Y, Z)] - [\Phi_1(X, Y), Z] - (-1)^{XY} [Y, \Phi_1(X, Z)] = 0 , \end{aligned} \quad (6)$$

for $n = 1$ is a condition on $\Phi_1 : \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$, which can be interpreted as a cocycle condition in the cochain complex $C^2(\mathfrak{g}; \mathfrak{g})$ to be defined below. A Φ_1 obeying equation (6) is said to be an *infinitesimal deformation*. Such an infinitesimal deformation is said to be *trivial*, if it is the result of a t -dependent change of basis; in other words, if Φ_1 is given by

$$\Phi_1(X, Y) = [X, B(Y)] - (-1)^{XY} [Y, B(X)] - B([X, Y]) , \quad (7)$$

for some even linear transformation $B : \mathfrak{g} \rightarrow \mathfrak{g}$. It is easy to check that such Φ_1 automatically obeys (6). Indeed, equation (7) says that Φ_1 is a coboundary in $C^2(\mathfrak{g}; \mathfrak{g})$. The space of (nontrivial) infinitesimal deformations is therefore the space of solutions Φ_1 of (6) factored by the space of Φ_1 given by (7), which can be reinterpreted as the cohomology group $H^2(\mathfrak{g}; \mathfrak{g})$ to

be defined below. The further equations in (5) for higher n give obstructions to integrating the infinitesimal deformation. They can be reinterpreted as a sequence of cohomology classes in $H^3(\mathfrak{g}; \mathfrak{g})$. In a nutshell, the tangent space to the moduli space of deformations of a Lie superalgebra \mathfrak{g} is given by $H^2(\mathfrak{g}; \mathfrak{g})$, whereas the obstructions to integrating a deformation along a direction in $H^2(\mathfrak{g}; \mathfrak{g})$ are given by a sequence of classes in $H^3(\mathfrak{g}; \mathfrak{g})$ and which are in the image of a squaring map $H^2(\mathfrak{g}; \mathfrak{g}) \rightarrow H^3(\mathfrak{g}; \mathfrak{g})$ described in [27]. For example, the equation for $n = 2$ says that the 3-cocycle

$$[\Phi_1, \Phi_1](X, Y, Z) := \Phi_1(X, \Phi_1(Y, Z)) - \Phi_1(\Phi_1(X, Y), Z) - (-1)^{XY} \Phi_1(Y, \Phi_1(X, Z)) \quad (8)$$

obtained by “squaring” Φ_1 should be a coboundary (of Φ_2), et cetera.

2.2. Cohomology. Lie superalgebra cohomology was introduced by Leites [28] and is reviewed in [29, §1.6]. It is a straight-forward extension of the better-known Lie algebra cohomology theory of Chevalley and Eilenberg [30].

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional real Lie superalgebra and let $\mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{M}_1$ be a \mathfrak{g} -module. We will let $X \cdot m$ denote the action of $X \in \mathfrak{g}$ on $m \in \mathfrak{M}$. We demand that the action preserve the parity, so that $\mathfrak{g}_\alpha \cdot \mathfrak{M}_\beta \subset \mathfrak{M}_{\alpha+\beta}$. Let $C^n(\mathfrak{g}; \mathfrak{M})$ denote the space of multilinear maps

$$f : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{n \text{ times}} \rightarrow \mathfrak{M} \quad (9)$$

satisfying the following skewsymmetry condition:

$$f(X_1, \dots, X_n) = -(-1)^{X_i X_{i+1}} f(X_1, \dots, X_{i-1}, X_{i+1}, X_i, X_{i+2}, \dots, X_n) . \quad (10)$$

The vector space $C^n(\mathfrak{g}; \mathfrak{M})$ of such maps is naturally \mathbb{Z}_2 -graded. We will let

$$C(\mathfrak{g}; \mathfrak{M}) = \bigoplus_{n=0}^{\infty} C^n(\mathfrak{g}; \mathfrak{M}) . \quad (11)$$

If $\mathfrak{h} < \mathfrak{g}$ is an ideal, then each vector space $C^n(\mathfrak{h}; \mathfrak{M})$ is naturally a \mathfrak{g} -module, where for all $Y \in \mathfrak{g}$, $f \in C^n(\mathfrak{h}; \mathfrak{M})$ and $X_1, \dots, X_n \in \mathfrak{h}$,

$$(Y \cdot f)(X_1, \dots, X_n) = Y \cdot f(X_1, \dots, X_n) - \sum_{i=1}^n (-1)^{Y(f+X_1+\cdots+X_{i-1})} f(X_1, \dots, [Y, X_i], \dots, X_n) . \quad (12)$$

We define the differential $d : C^n(\mathfrak{g}; \mathfrak{M}) \rightarrow C^{n+1}(\mathfrak{g}; \mathfrak{M})$ as follows. If $m \in C^0(\mathfrak{g}; \mathfrak{M}) = \mathfrak{M}$,

$$(dm)(X) = (-1)^{Xm} X \cdot m , \quad (13)$$

and if $f \in C^{n>0}(\mathfrak{g}; \mathfrak{M})$,

$$\begin{aligned} (df)(X_0, X_1, \dots, X_n) &= \sum_{i=0}^n (-1)^{i+X_i(f+X_0+\cdots+X_{i-1})} X_i \cdot f(X_0, \dots, \widehat{X_i}, \dots, X_n) \\ &+ \sum_{0 \leq i < j \leq n} (-1)^{i+j+(X_i+X_j)(X_0+\cdots+X_{i-1})+X_j(X_{i+1}+\cdots+X_{j-1})} \\ &\quad \times f([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_n) . \end{aligned} \quad (14)$$

Notice that d has zero parity. It obeys $d^2 = 0$ and it is \mathfrak{g} -equivariant, so that $X \cdot df = d(X \cdot f)$ for all $X \in \mathfrak{g}$ and $f \in C^m(\mathfrak{g}; \mathfrak{M})$. For every $X \in \mathfrak{g}$ we define a derivation $\iota_X : C^m(\mathfrak{g}; \mathfrak{M}) \rightarrow C^{m-1}(\mathfrak{g}; \mathfrak{M})$ by

$$(\iota_X f)(X_1, \dots, X_{n-1}) = (-1)^{fX} f(X, X_1, \dots, X_{n-1}) . \quad (15)$$

It follows easily that

$$\iota_X(Y \cdot f) - (-1)^{XY} Y \cdot \iota_X f = \iota_{[X,Y]} f \quad (16)$$

and also that the Cartan formula holds

$$\iota_X df + d\iota_X f = X \cdot f . \quad (17)$$

Let (X_a, X_i) and (m_A, m_I) denote homogeneous bases for $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{M}_1$, respectively. Here and in what follows we will adhere to the summation convention. Doing so, we have

$$\begin{aligned} X_a \cdot m_A &= K_{aA}^B m_B & \text{and} & & X_i \cdot m_A &= K_{iA}^I m_I \\ X_a \cdot m_I &= K_{aI}^J m_J & & & X_i \cdot m_I &= K_{iI}^A m_A , \end{aligned} \quad (18)$$

and also

$$[X_a, X_b] = f_{ab}^c X_c , \quad [X_a, X_i] = f_{ai}^j X_j \quad \text{and} \quad [X_i, X_j] = f_{ij}^a X_a . \quad (19)$$

Let (θ^a, θ^i) denote the canonical dual basis for $\mathfrak{g}^* = \mathfrak{g}_0^* \oplus \mathfrak{g}_1^*$. The following rules, together with the fact that d is a derivation, suffice to compute the differential on any cochain in $C^m(\mathfrak{g}; \mathfrak{M})$:

$$\begin{aligned} d\theta^a &= -\frac{1}{2} f_{bc}^a \theta^b \wedge \theta^c + \frac{1}{2} f_{ij}^a \theta^i \wedge \theta^j & dm_A &= \theta^a \otimes K_{aA}^B m_B - \theta^i \otimes K_{iA}^I m_I \\ d\theta^i &= -f_{aj}^i \theta^a \wedge \theta^j & dm_I &= \theta^a \otimes K_{aI}^J m_J - \theta^i \otimes K_{iI}^A m_A . \end{aligned} \quad (20)$$

Notice that our convention for \wedge is that $\alpha \wedge \beta = -(-1)^{\alpha\beta} \beta \wedge \alpha$, so that it is superskewsymmetric; in particular, $\theta^i \wedge \theta^j = \theta^j \wedge \theta^i$. As a check of these formulae, it may be shown that the differential of the identity map $\mathfrak{g} \rightarrow \mathfrak{g}$, thought of as the 1-cochain $\theta^a \otimes X_a - \theta^i \otimes X_i \in C^1(\mathfrak{g}; \mathfrak{g})$, is the 2-cochain

$$\frac{1}{2} f_{ab}^c \theta^a \wedge \theta^b \otimes X_c - f_{ai}^j \theta^a \wedge \theta^i \otimes X_j - \frac{1}{2} f_{ij}^a \theta^i \wedge \theta^j \otimes X_a \in C^2(\mathfrak{g}; \mathfrak{g}) \quad (21)$$

corresponding to the Lie bracket.

2.3. Hochschild–Serre factorisation theorem. A fundamental tool in computing these cohomology groups is the Hochschild–Serre spectral sequence [31] which exploits the existence of a semisimple factor \mathfrak{s} of \mathfrak{g} in order to reduce the calculation of the cohomology to that of the much smaller subcomplex of \mathfrak{s} -invariants. This method was used in [32] to calculate the possible Lie algebra deformations of the Galilean algebras. The superalgebra version of this theorem is discussed in [29, §1.6.5]; although it has also appeared in [33, 34]. In [33] it was used in order to compute the possible deformations of the four-dimensional Poincaré superalgebra and $\mathfrak{osp}(4|2)$; although the deformed Poincaré algebra in that paper is actually incorrect. A correct calculation of the unique deformation [35] of the four-dimensional Poincaré superalgebra appears in [34], which also contains a fuller treatment of the Hochschild–Serre spectral sequence. In a nutshell, the theorem allows us to work covariantly with respect to any semisimple subalgebra of the Lie superalgebra in question.

More precisely, let \mathfrak{g} be a finite-dimensional real Lie superalgebra and let \mathfrak{M} denote a \mathfrak{g} -module. Let $I < \mathfrak{g}$ be an ideal such that $\mathfrak{s} := \mathfrak{g}/I$ is a semisimple Lie algebra. Then the

factorisation theorem of Hochschild–Serre states that

$$H^n(\mathfrak{g}; \mathfrak{M}) \cong \bigoplus_{i=0}^n (H^{n-i}(\mathfrak{s}) \otimes H^i(I; \mathfrak{M})^{\mathfrak{s}}) , \quad (22)$$

where $H^\bullet(I; \mathfrak{M})^{\mathfrak{s}}$ is the cohomology of the subcomplex $C^\bullet(I; \mathfrak{M})^{\mathfrak{s}}$ of \mathfrak{s} -invariant cochains and $H^\bullet(\mathfrak{s})$ is the cohomology with values in the trivial one-dimensional module. Using the Whitehead lemma, $H^j(\mathfrak{s}) = 0$ for $j = 1, 2$, and the fact that $H^0(\mathfrak{s}) \cong \mathbb{R}$, the above direct sum simplifies to

$$H^n(\mathfrak{g}; \mathfrak{M}) \cong H^n(I; \mathfrak{M})^{\mathfrak{s}} \oplus \bigoplus_{i=0}^{n-3} (H^{n-i}(\mathfrak{s}) \otimes H^i(I; \mathfrak{M})^{\mathfrak{s}}) . \quad (23)$$

In particular, we have that

$$H^1(\mathfrak{g}; \mathfrak{g}) \cong H^1(I; \mathfrak{g})^{\mathfrak{s}} \quad \text{and} \quad H^2(\mathfrak{g}; \mathfrak{g}) \cong H^2(I; \mathfrak{g})^{\mathfrak{s}} , \quad (24)$$

whereas

$$H^3(\mathfrak{g}; \mathfrak{g}) \cong H^3(I; \mathfrak{g})^{\mathfrak{s}} \oplus H^3(\mathfrak{s}) \otimes \mathfrak{z} , \quad (25)$$

where $\mathfrak{z} = \mathfrak{g}^{\mathfrak{g}}$ is the centre of \mathfrak{g} . Of course, the full strength of this theorem is only ever felt if \mathfrak{g} admits a sufficiently large semisimple factor \mathfrak{s} .

3. RIGIDITY OF THE POINCARÉ SUPERALGEBRA

As a first calculation, let us take \mathfrak{g} to be the 11-dimensional Poincaré superalgebra, which is the Killing superalgebra of the Minkowski background of 11-dimensional supergravity. We take I to be the supertranslation ideal consisting of the momentum generators and the supercharges. The semisimple factor is the Lorentz subalgebra $\mathfrak{s} \cong \mathfrak{so}(10, 1)$. As \mathfrak{s} -modules, $I = V \oplus \Delta$ and $\mathfrak{g} = V \oplus \Lambda^2 V \oplus \Delta$, where V is the 11-dimensional vector representation and Δ is the real 32-dimensional irreducible representation of spinors.

It will prove convenient *not* to identify I and $V \oplus \Delta$. We will let P, Q denote the isomorphisms between V and Δ and the corresponding subspaces of I , and similarly we will let $L : \Lambda^2 V \rightarrow \mathfrak{so}(V)$ denote the natural isomorphism.

Let e_a be a basis for V and ε_i be a basis for Δ . We will choose an action of the Clifford algebra $\text{Cl}(V)$ on Δ once and for all. Following the time-honoured tradition, the image of e_a under the embedding $V \rightarrow \text{Cl}(V)$ will be denoted γ_a . Our conventions are $\gamma_a \gamma_b + \gamma_b \gamma_a = +2\eta_{ab} \mathbf{1}$, with η_{ab} mostly plus. More traditionally still, we will let $\gamma_{ab\dots c}$ denote the image of $e_a \wedge e_b \wedge \dots \wedge e_c$ under the vector space isomorphism $\Lambda V \xrightarrow{\cong} \text{Cl}(V)$.

The corresponding bases for I are $P_a := P(e_a)$ and $Q_i = Q(\varepsilon_i)$. We will let e^a be the canonical dual basis of V^* . Because \mathfrak{s} leaves invariant the Minkowski metric $\eta \in S^2 V^*$, we may identify V and V^* by “raising/lowering indices” with η and our notation reflects this. Similarly we let ε^i denote the canonical dual basis for Δ^* , where we may again identify Δ and Δ^* using the \mathfrak{s} -invariant symplectic form on Δ . Letting P and Q also stand for the isomorphisms of V^* and Δ^* with the corresponding subspaces of I^* , we will let $P^a = P(e^a)$ and $Q^i = Q(\varepsilon^i)$. Finally we will also let $L_{ab} := L(e_a \wedge e_b)$, for $a < b$, define a basis for \mathfrak{s} .

The Poincaré superalgebra consists of a Lorentz subalgebra spanned by the L_{ab} and in addition

$$\begin{aligned} [L_{ab}, Q_i] &= \tfrac{1}{2} \gamma_{ab} \cdot Q_i \\ [L_{ab}, P_c] &= \eta_{bc} P_a - \eta_{ac} P_b \\ [Q_i, Q_j] &= \gamma_{ij}^a P_a , \end{aligned} \quad (26)$$

where

$$\gamma_{ab} \cdot Q_i = Q(\gamma_{ab} \cdot \varepsilon_i) = Q_j(\gamma_{ab})^j_i , \quad (27)$$

and similarly for the action of any other element in the Clifford algebra $\text{Cl}(V)$, and

$$\gamma_{ij}^a := \langle \varepsilon_i, \gamma^a \cdot \varepsilon_j \rangle , \quad (28)$$

where $\langle -, - \rangle$ is the \mathfrak{s} -invariant symplectic structure on Δ .

Let us investigate the subcomplex $C^\bullet := C^\bullet(I; \mathfrak{g})^{\mathfrak{s}}$ of Lorentz-invariant cochains in $C^\bullet(I; \mathfrak{g})$ in low dimension. For applications to the theory of Lie superalgebra deformations we are interested only in *even* cochains; however this is not a restriction because of the representation-theoretic “spin statistics” theorem, which states that there are no Lorentz-invariant maps between “fermionic” and “bosonic” representations.

There are no Lorentz-invariant 0-cochains, since \mathfrak{g} contains no Lorentz scalars. The even cochains in $C^1(I; \mathfrak{g})$ are maps $V \rightarrow V \oplus \Lambda^2 V$ and $\Delta \rightarrow \Delta$. Since V and Δ are irreducible representations, Schur’s lemma says that the only equivariant maps are multiples of the identity maps $V \rightarrow V$ and $\Delta \rightarrow \Delta$. Therefore, a basis for C^1 is given by the vectors

$$P^a \otimes P_a \quad \text{and} \quad Q^i \otimes Q_i . \quad (29)$$

The even 2-cochains are maps of the form $\Lambda^2 V \rightarrow V \oplus \Lambda^2 V$, $V \otimes \Delta \rightarrow \Delta$ and $S^2 \Delta \rightarrow V \oplus \Lambda^2 V$. Lorentz invariance is again very restrictive and there is a four-dimensional subspace of equivariant maps which is spanned by the identity map $\Lambda^2 V \rightarrow \Lambda^2 V$, Clifford multiplication $V \otimes \Delta \rightarrow \Delta$ and the spinor squaring maps $S^2 \Delta \rightarrow V$ and $S^2 \Delta \rightarrow \Lambda^2 V$. A basis for C^2 is given by

$$\begin{aligned} P^a \wedge P^b \otimes L_{ab} & \quad Q^i \wedge Q^j \otimes \gamma_{ij}^a P_a \\ P^a \wedge Q^i \otimes (\gamma_a \cdot Q_i) & \quad Q^i \wedge Q^j \otimes \gamma_{ij}^{ab} L_{ab} . \end{aligned} \quad (30)$$

The even 3-cochains are given by maps of the form $\Lambda^3 V \rightarrow V \oplus \Lambda^2 V$, $\Lambda^2 V \otimes \Delta \rightarrow \Delta$, $V \otimes S^2 \Delta \rightarrow V \oplus \Lambda^2 V$ and $S^3 \Delta \rightarrow \Delta$. The Lorentz-equivariant maps are given in terms of the natural operations. There is no Lorentz-equivariant map $\Lambda^3 V \rightarrow V \oplus \Lambda^2 V$ because all three representations V , $\Lambda^2 V$ and $\Lambda^3 V$ are irreducible and non-equivalent. The only equivariant map $\Lambda^2 V \otimes \Delta \rightarrow \Delta$ is the spin representation, for which a representative cochain is given by

$$P^a \wedge P^b \wedge Q^i \otimes \gamma_{ab} \cdot Q_i . \quad (31)$$

The equivariant maps $V \otimes S^2 \Delta \rightarrow V \oplus \Lambda^2 V$ are given by the compositions

$$V \otimes S^2 \Delta \longrightarrow V \otimes (V \oplus \Lambda^2 V) \xrightarrow{\wedge \oplus \iota} \Lambda^2 V \oplus V , \quad (32)$$

whose representative cochains are

$$P^a \wedge Q^i \wedge Q^j \otimes \gamma_{ij}^b L_{ab} \quad \text{and} \quad P^a \wedge Q^i \wedge Q^j \otimes (\gamma_a^b)_{ij} P_b . \quad (33)$$

Finally, the equivariant maps $S^3 \Delta \rightarrow \Delta$ have the following representative cochains:

$$Q^i \wedge Q^j \wedge Q^k \otimes \gamma_{ij}^a \gamma_a \cdot Q_k \quad \text{and} \quad Q^i \wedge Q^j \wedge Q^k \otimes \gamma_{ij}^{ab} \gamma_{ab} \cdot Q_k . \quad (34)$$

We see that C^3 is therefore five dimensional.

The differential d in the invariant subcomplex

$$0 \longrightarrow C^1 \xrightarrow{d} C^2 \xrightarrow{d} C^3 \longrightarrow \dots \quad (35)$$

is defined by its action on the elements of I^* and of \mathfrak{g} as an I -module; that is,

$$\begin{aligned} dP^a &= \frac{1}{2}\gamma_{ij}^a Q^i \wedge Q^j \\ dQ^i &= 0 \\ dP_a &= 0 \\ dQ_i &= -\gamma_{ij}^a Q^j \otimes P_a \\ dL_{ab} &= \eta_{ac} P^c \otimes P_b - \eta_{bc} P^c \otimes P_a + \frac{1}{2} Q^i \otimes \gamma_{ab} \cdot Q_i . \end{aligned} \tag{36}$$

Acting on the 1-cochains, we see that

$$\begin{aligned} d(P^a \otimes P_a) &= \frac{1}{2}\gamma_{ij}^a Q^i \wedge Q^j \otimes P_a \\ d(Q^i \otimes Q_i) &= \gamma_{ij}^a Q^i \wedge Q^j \otimes P_a . \end{aligned} \tag{37}$$

Thus $2P^a \otimes P_a - Q^i \otimes Q_i$ is a cocycle, whence, in the absence of any coboundaries, we conclude that $H^1(\mathfrak{g}; \mathfrak{g}) \cong \mathbb{R}$. This corresponds to an outer derivation of \mathfrak{g} given by dilatations. The corresponding extension is obtained by replacing $\mathfrak{so}(10, 1)$ by $\mathfrak{co}(10, 1)$ acting on \mathfrak{g} in such that way that L_{ab}, Q_i, P_a have weights 0, 1 and 2, respectively.

We also learn that the 2-cochain $\gamma_{ij}^a Q^i \wedge Q^j \otimes P_a$ is a coboundary. The differential of the remaining three 2-cochains are

$$\begin{aligned} d(P^a \wedge P^b \otimes L_{ab}) &= \gamma_{ij}^a Q^i \wedge Q^j \wedge P^b \otimes L_{ab} + \frac{1}{2} P^a \wedge P^b \wedge Q^i \otimes \gamma_{ab} \cdot Q_i , \\ d(P^a \wedge Q^i \otimes (\gamma_a \cdot Q_i)) &= (\gamma_a^b)_{ij} P^a \wedge Q^i \wedge Q^j \otimes P_b + \frac{1}{2} \gamma_{ij}^a Q^i \wedge Q^j \wedge Q^k \otimes \gamma_a \cdot Q_k , \\ d(\gamma_{ij}^{ab} Q^i \wedge Q^j \otimes L_{ab}) &= 2(\gamma_a^b)_{ij} P^a \wedge Q^i \wedge Q^j \otimes P_b + \frac{1}{2} \gamma_{ij}^{ab} Q^i \wedge Q^j \wedge Q^k \otimes \gamma_{ab} \cdot Q_k . \end{aligned} \tag{38}$$

The only possible cocycle would be

$$\Psi := 2P^a \wedge Q^i \otimes (\gamma_a \cdot Q_i) - \gamma_{ij}^{ab} Q^i \wedge Q^j \otimes L_{ab} , \tag{39}$$

whose differential is

$$d\Psi = Q^i \wedge Q^j \wedge Q^k \otimes \left(\frac{1}{2} \gamma_{ij}^{ab} \gamma_{ab} \cdot Q_k - \gamma_{ij}^a \gamma_a \cdot Q_k \right) . \tag{40}$$

By the usual polarisation identity, which says that if $P \in S^3 V^*$ and $p(\mathbf{v}) = P(\mathbf{v}, \mathbf{v}, \mathbf{v})$ is the associated cubic form, then

$$\begin{aligned} P(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &= \frac{1}{6} (p(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) - p(\mathbf{v}_1 + \mathbf{v}_2) - p(\mathbf{v}_1 + \mathbf{v}_3) \\ &\quad - p(\mathbf{v}_2 + \mathbf{v}_3) + p(\mathbf{v}_1) + p(\mathbf{v}_2) + p(\mathbf{v}_3)) , \end{aligned} \tag{41}$$

equation (40) would vanish if and only if for all spinors $\psi \in \Delta$,

$$\langle \psi, \gamma^a \cdot \psi \rangle \gamma_a \cdot \psi - \frac{1}{2} \left\langle \psi, \gamma^{ab} \cdot \psi \right\rangle \gamma_{ab} \cdot \psi \stackrel{?}{=} 0 . \tag{42}$$

This is a Lorentz-covariant equation, whence it is zero for a $\psi \in \Delta$ it will be zero for every other spinor in its Lorentz orbit. There are two possible orbits (apart from the trivial orbit consisting of the zero spinor, for which this equation is trivially satisfied). The above identity holds for the small orbit consisting of spinors whose Dirac current is null, but it does not hold for the generic orbit consisting of spinors whose Dirac current is timelike. Indeed, for all $\psi \in \Delta$, one finds that

$$\langle \psi, \gamma^a \cdot \psi \rangle \gamma_a \cdot \psi + \frac{1}{10} \left\langle \psi, \gamma^{ab} \cdot \psi \right\rangle \gamma_{ab} \cdot \psi = 0 ; \tag{43}$$

although for ψ in the small orbit both terms vanish separately. Therefore we conclude that $H^2(\mathfrak{g}; \mathfrak{g}) = 0$ and the 11-dimensional Poincaré superalgebra is rigid.

This is in sharp contrast with the four-dimensional case. As shown in [33, 34], the four-dimensional superalgebra admits a deformation [35] whose bosonic subalgebra is the isometry algebra of anti de Sitter spacetime.

This result is consistent with the fact that the Minkowski vacuum does not receive M-theoretic corrections, which follows from the observation that corrections to the equations of motions come in the shape of polynomials of the curvature and the field strength, both of which vanish for this background.

It is well-known that the Minkowski background arises as various limits of the other maximally supersymmetric backgrounds. These limits are known to contract the Killing superalgebra, whence one might expect to discover deformations of the Killing superalgebra of the Minkowski background which reverse these contractions and hence one might be puzzled by the rigidity found above. The solution to the puzzle is to notice that the dimension of the Killing superalgebras of the Freund–Rubin [36, 37] or Kowalski-Glikman [38] backgrounds is $(38|32)$, whereas that of the Minkowski background is $(66|32)$. There are subalgebras of the Poincaré superalgebra, namely the image of the contractions of the Killing superalgebras of the other maximally supersymmetric backgrounds, which do admit deformations, but the full superalgebra does not. This shows that one must exercise care when concluding the existence or otherwise of corrected supergravity backgrounds based solely on the existence of deformations of the corresponding Killing superalgebras.

On the other hand, the Killing superalgebra of the Kowalski-Glikman background does have deformations, which are the Killing superalgebras of the Freund–Rubin backgrounds; although as we will now argue, the superalgebras of the latter backgrounds are actually rigid.

3.1. Rigidity of the Freund–Rubin superalgebras. As reviewed for example in [5], the Killing superalgebras of the Freund–Rubin backgrounds $\text{AdS}_4 \times S^7$ and $\text{AdS}_7 \times S^4$ are $\mathfrak{osp}(8|4)$ and $\mathfrak{osp}(6, 2|4)$, respectively, which are real forms of the complex Lie superalgebra of type $D(4, 2)$ in the notation of Kac [39], whose Killing form is nondegenerate. The proof of the rigidity of semisimple Lie algebras uses the nondegeneracy of the Killing form to construct a chain homotopy, whence we expect that for Lie superalgebras with nondegenerate Killing form, the same result should obtain. Indeed, this has already been shown in [40], from where we deduce the rigidity of the Killing superalgebras of the Freund–Rubin backgrounds. This agrees with the fact that these backgrounds do not receive quantum corrections [41].

4. SUPERALGEBRA DEFORMATIONS OF BRANE BACKGROUNDS

In this section we detail the calculations of $H^2(\mathfrak{g}; \mathfrak{g})$ for the Killing superalgebras of the $\frac{1}{2}$ -BPS maximally symmetric M2- and M5-brane backgrounds. The Killing superalgebras are subsuperalgebras of the one for the Minkowski background, to which the branes are asymptotic.

4.1. A deformation of the M2-brane Killing superalgebra. The Killing superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of the M2-brane is the subalgebra of the Poincaré superalgebra defined as follows. First we split the 11-dimensional lorentzian vector space $V = W \oplus W^\perp$, where W is three-dimensional lorentzian. The subalgebra of $\mathfrak{so}(V)$ which preserves this split is $\mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp)$. Then $\mathfrak{g}_0 = \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp) \oplus W$. The odd part of the superalgebra is the subspace Δ of the $\mathfrak{so}(V)$ spinor module $\Delta(V)$ consisting of those spinors ψ for which $\nu_W \cdot \psi = \psi$, where ν is the

element in the Clifford algebra representing the volume form. As an $\mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp)$ -module, this is $\Delta(W) \otimes \Delta(W^\perp)_-$, where the chirality condition on the $\mathfrak{so}(W^\perp)$ -spinor comes about because \mathfrak{g}_1 is the subspace of the irreducible $C\ell(V)$ -module consisting of spinors ψ for which $\nu_V \cdot \psi = -\psi$ and $\nu_W \cdot \psi = \psi$, whence $\nu_{W^\perp} \cdot \psi = -\psi$. The resulting Lie superalgebra has dimension $(34|16)$.

In order to apply the Hochschild–Serre factorisation theorem, we will choose I to be the supertranslation ideal, which is isomorphic to $W \oplus \Delta$, so that $\mathfrak{s} = \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp)$. We will let e_μ and e_a be a basis for W and W^\perp , respectively, and let ε_i be a basis for Δ . Unlike in the previous section, here i only goes from 1 to 16. We will let P_μ and Q_i denote the corresponding bases for I and $L_{\mu\nu}$ and L_{ab} the corresponding bases for \mathfrak{s} . As before we will let P^μ and Q^i denote the bases for I^* canonically dual to P_μ and Q_i , respectively.

In this basis, the Lie brackets are

$$\begin{aligned} [L_{\mu\nu}, Q_i] &= \frac{1}{2} \gamma_{\mu\nu} \cdot Q_i \\ [L_{ab}, Q_i] &= \frac{1}{2} \gamma_{ab} \cdot Q_i \\ [L_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \\ [Q_i, Q_j] &= \gamma_{ij}^\mu P_\mu \end{aligned} \tag{44}$$

in addition to the ones of \mathfrak{s} .

There are no \mathfrak{s} -invariant elements in \mathfrak{g} , whence there are no invariant 0-cochains. The space C^1 of invariant 1-cochains is three-dimensional, spanned by the identity maps $W \rightarrow W$ and $\Delta \rightarrow \Delta$, as well as the natural \mathfrak{s} -equivariant isomorphism $W \rightarrow \Lambda^2 W$ induced from Hodge duality. The representative cochains are

$$P^\mu \otimes P_\mu, \quad Q^i \otimes Q_i \quad \text{and} \quad \varepsilon_\rho^{\mu\nu} P^\rho \otimes L_{\mu\nu}. \tag{45}$$

The space C^2 of invariant 2-cochains is six-dimensional, spanned by the Hodge duality map $\Lambda^2 W \rightarrow W$ and the identity map $\Lambda^2 W \rightarrow \Lambda^2 W$, as well as Clifford product $W \otimes \Delta \rightarrow \Delta$, and the three squaring maps $S^2 \Delta \rightarrow W \oplus \Lambda^2 W \oplus \Lambda^2 W^\perp$. The representative cochains are

$$\begin{aligned} \varepsilon_{\mu\nu}^\rho P^\mu \wedge P^\nu \otimes P_\rho & \quad \gamma_{ij}^\mu Q^i \wedge Q^j \otimes P_\mu \\ P^\mu \wedge P^\nu \otimes L_{\mu\nu} & \quad \gamma_{ij}^{\mu\nu} Q^i \wedge Q^j \otimes L_{\mu\nu} \\ P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i & \quad \gamma_{ij}^{ab} Q^i \wedge Q^j \otimes L_{ab}. \end{aligned} \tag{46}$$

Notice the term $\varepsilon_{\mu\nu\rho} P^\mu \wedge Q^i \otimes \gamma^{\nu\rho} \cdot Q_i$ is omitted, due to the fact that $\nu_W \cdot Q_i = Q_i$, whence $\gamma_\mu \cdot Q_i$ and $\varepsilon_{\mu\nu\rho} \gamma^{\nu\rho} \cdot Q_i$ are proportional. Indeed, $\varepsilon_{\mu\nu\rho} \gamma^{\nu\rho} \cdot Q_i = -\gamma_{\mu\nu} \cdot Q_i$ and $\frac{1}{2} \varepsilon_{\mu\nu\rho} \gamma^{\nu\rho} \cdot Q_i = \gamma_\mu \cdot Q_i$.

The space C^3 of invariant 3-cochains is four-dimensional. It is spanned by the following natural maps:

- $\Lambda^2 W \otimes \Delta \rightarrow \Delta$, given by the $\mathfrak{so}(W)$ action;
- $W \otimes S^2 \Delta \rightarrow W \oplus \Lambda^2 W$, given by the squaring map on spinors;
- $S^3 \Delta \rightarrow \Delta$, given by the composition

$$S^3 \Delta \longrightarrow W \otimes \Delta \longrightarrow \Delta, \tag{47}$$

where the first map is the squaring of the spinors and the second is made out of Clifford multiplication by W on Δ .

The absence of any $\Lambda^2 W^\perp$ in the above cochains should not have gone unnoticed by the attentive reader. It is not hard to show that there is no equivariant map $W \otimes S^2 \Delta \rightarrow \Lambda^2 W^\perp$,

since the corresponding cochain $\langle \varepsilon_i, \gamma_\mu \cdot \gamma^{ab} \cdot \varepsilon_j \rangle P^\mu \wedge Q^i \wedge Q^j \otimes L_{ab}$ vanishes because of the skew-symmetry (in ij) of $\langle \varepsilon_i, \gamma_\mu \cdot \gamma^{ab} \cdot \varepsilon_j \rangle$. Similarly, the composition $S^3\Delta \rightarrow \Lambda^2 W^\perp \otimes \Delta \rightarrow \Delta$ can be written as a linear combination of the composition $S^3\Delta \rightarrow W \otimes \Delta \rightarrow \Delta$, by virtue of (43). Indeed, unpacking (43) under $\mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp)$, we find

$$\begin{aligned} & \langle \psi, \gamma^\mu \cdot \psi \rangle \gamma_\mu \cdot \psi + \langle \psi, \gamma^a \cdot \psi \rangle \gamma_a \cdot \psi + \frac{1}{10} \langle \psi, \gamma^{ab} \cdot \psi \rangle \gamma_{ab} \cdot \psi \\ & + \frac{1}{10} \langle \psi, \gamma^{\mu\nu} \cdot \psi \rangle \gamma_{\mu\nu} \cdot \psi + \frac{1}{5} \langle \psi, \gamma^a \cdot \gamma^\mu \cdot \psi \rangle \gamma_a \cdot \gamma_\mu \cdot \psi = 0 . \end{aligned} \quad (48)$$

The condition $\nu_W \cdot \psi = \psi$ says that

$$\langle \psi, \gamma^{\mu\nu} \cdot \psi \rangle \gamma_{\mu\nu} \cdot \psi = -2 \langle \psi, \gamma^\mu \cdot \psi \rangle \gamma_\mu \cdot \psi , \quad (49)$$

This shows why we did not list the composition $S^3\Delta \rightarrow \Lambda^2 W \otimes \Delta \rightarrow \Delta$ among the maps above.

In addition, the self-adjointness of ν_W relative to the spinor inner product and the relations $\nu_W \cdot w = w \cdot \nu_W$ for $w \in W$ and $\nu_W \cdot v = -v \cdot \nu_W$ for $v \in W^\perp$, means that $\langle \psi, \gamma^a \cdot \psi \rangle = 0$ and $\langle \psi, \gamma^a \cdot \gamma^\mu \cdot \psi \rangle = 0$, whence the identity (43) can be rewritten as

$$\langle \psi, \gamma^{ab} \cdot \psi \rangle \gamma_{ab} \cdot \psi = -8 \langle \psi, \gamma^\mu \cdot \psi \rangle \gamma_\mu \cdot \psi . \quad (50)$$

By the usual polarisation trick, this rules out the existence of the extra cochains involving $\Lambda^2 W^\perp$.

An explicit basis for the invariant 3-cochains is given by

$$\begin{aligned} P^\mu \wedge P^\nu \wedge Q^i \otimes \gamma_{\mu\nu} \cdot Q_i & \quad (\gamma_\mu^\nu)_{ij} P^\mu \wedge Q^i \wedge Q^j \otimes P_\nu \\ \gamma_{ij}^\mu Q^i \wedge Q^j \wedge Q^k \otimes \gamma_\mu \cdot Q_k & \quad (\gamma^\mu)_{ij} P^\nu \wedge Q^i \wedge Q^j \otimes L_{\mu\nu} . \end{aligned} \quad (51)$$

The differential of the invariant subcomplex (C^\bullet, d) is defined by its action on the elements of I^* and of \mathfrak{g} as an I -module; that is,

$$\begin{aligned} dP^\mu &= \frac{1}{2} \gamma_{ij}^\mu Q^i \wedge Q^j \\ dQ^i &= 0 \\ dP_\mu &= 0 \\ dQ_i &= -\gamma_{ij}^\mu Q^j \otimes P_\mu \\ dL_{\mu\nu} &= \eta_{\mu\rho} P^\rho \otimes P_\nu - \eta_{\nu\rho} P^\rho \otimes P_\mu + \frac{1}{2} Q^i \otimes \gamma_{\mu\nu} \cdot Q_i \\ dL_{ab} &= \frac{1}{2} Q^i \otimes \gamma_{ab} \cdot Q_i . \end{aligned} \quad (52)$$

We now compute the differential $d : C^1 \rightarrow C^2$:

$$\begin{aligned} d(P^\mu \otimes P_\mu) &= \frac{1}{2} \gamma_{ij}^\mu Q^i \wedge Q^j \otimes P_\mu \\ d(Q^i \otimes Q_i) &= \gamma_{ij}^\mu Q^i \wedge Q^j \otimes P_\mu \end{aligned} \quad (53)$$

$$d(\varepsilon_\rho^{\mu\nu} P^\rho \otimes L_{\mu\nu}) = -2\varepsilon_{\mu\nu}^\rho P^\mu \wedge P^\nu \otimes P_\rho - P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i - \frac{1}{2} \gamma_{ij}^{\mu\nu} Q^i \wedge Q^j \otimes L_{\mu\nu} .$$

We see that there is precisely one cocycle: $2P^\mu \otimes P_\mu - Q^i \otimes Q_i$, whence $H^1(\mathfrak{g}; \mathfrak{g}) \cong \mathbb{R}$ in the absence of any coboundaries. As before, this outer derivation can be interpreted as dilatations with the same weights as in the case of the Poincaré superalgebra. Since $\dim C^1 = 3$ and the space Z^1 of 1-cocycles has dimension 1, we see that the space B^2 of 2-cocycles has dimension 2.

To compute the differential $d : C^2 \rightarrow C^3$, we can recycle many of the results from the similar calculation in §3. From the computation above of $d : C^1 \rightarrow C^2$, we learn that $\gamma_{ij}^\mu Q^i \wedge Q^j \otimes P_\mu$ is a coboundary. For the remaining cochains one obtains

$$\begin{aligned}
d(\varepsilon_{\mu\nu}^\rho P^\mu \wedge P^\nu \otimes P_\rho) &= -(\gamma_\mu^\nu)_{ij} P^\mu \wedge Q^i \wedge Q^j \otimes P_\nu , \\
d(P^\mu \wedge P^\nu \otimes L_{\mu\nu}) &= (\gamma^\mu)_{ij} Q^i \wedge Q^j \wedge P^\nu \otimes L_{\mu\nu} + \frac{1}{2} P^\mu \wedge P^\nu \wedge Q^i \otimes \gamma_{\mu\nu} \cdot Q_i , \\
d(P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i) &= (\gamma_\mu^\nu)_{ij} P^\mu \wedge Q^i \wedge Q^j \otimes P_\nu + \frac{1}{2} \gamma_{ij}^\mu Q^i \wedge Q^j \wedge Q^k \otimes \gamma_\mu \cdot Q_k , \\
d(\gamma_{ij}^{\mu\nu} Q^i \wedge Q^j \otimes L_{\mu\nu}) &= 2(\gamma_\mu^\nu)_{ij} P^\mu \wedge Q^i \wedge Q^j \otimes P_\nu + \frac{1}{2} \gamma_{ij}^{\mu\nu} Q^i \wedge Q^j \wedge Q^k \otimes \gamma_{\mu\nu} \cdot Q_k , \\
d(\gamma_{ij}^{ab} Q^i \wedge Q^j \otimes L_{ab}) &= \frac{1}{2} \gamma_{ij}^{ab} Q^i \wedge Q^j \wedge Q^k \otimes \gamma_{ab} \cdot Q_k \\
&= -4\gamma_{ij}^\mu Q^i \wedge Q^j \wedge Q^k \otimes \gamma_\mu \cdot Q_k .
\end{aligned} \tag{54}$$

It is easy to construct a basis for the space Z^2 of cocycles:

$$\begin{aligned}
&\gamma_{ij}^\mu Q^i \wedge Q^j \otimes P_\mu \\
&\gamma_{ij}^{\mu\nu} Q^i \wedge Q^j \otimes L_{\mu\nu} + 2P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i + 4\varepsilon_{\mu\nu}^\rho P^\mu \wedge P^\nu \otimes P_\rho \\
&\gamma_{ij}^{ab} Q^i \wedge Q^j \otimes L_{ab} + 8P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i + 8\varepsilon_{\mu\nu}^\rho P^\mu \wedge P^\nu \otimes P_\rho .
\end{aligned} \tag{55}$$

Thus $\dim Z^2 = 3$. Since $\dim B^2 = 2$, with basis

$$\begin{aligned}
&\gamma_{ij}^\mu Q^i \wedge Q^j \otimes P_\mu \\
&2\varepsilon_{\mu\nu}^\rho P^\mu \wedge P^\nu \otimes P_\rho - P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i + \frac{1}{2} \gamma_{ij}^{\mu\nu} Q^i \wedge Q^j \otimes L_{\mu\nu} ,
\end{aligned} \tag{56}$$

we see that $\dim H^2(\mathfrak{g}; \mathfrak{g}) = 1$ and this allows us to conclude that there is an infinitesimal deformation of the M2 Killing superalgebra, with representative cocycle¹

$$\gamma_{ij}^{ab} Q^i \wedge Q^j \otimes L_{ab} + 8P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i + 8\varepsilon_{\mu\nu}^\rho P^\mu \wedge P^\nu \otimes P_\rho . \tag{57}$$

In order to determine whether this deformation is integrable, let us investigate the obstruction space $H^3(\mathfrak{g}; \mathfrak{g})$. Since $\dim Z^2 = 3$ and $\dim C^2 = 6$, it follows that $\dim B^3 = 3$. As $\dim C^4 = 4$, this means that $\dim H^3(I; \mathfrak{g})^s \leq 1$ with equality if and only if $d : C^3 \rightarrow C^4$ is the zero map. A simple calculation shows that, for instance,

$$\begin{aligned}
d(P^\mu \wedge P^\nu \wedge Q^i \otimes \gamma_{\mu\nu} \cdot Q_i) &= \gamma_{ij}^\mu P^\nu \wedge Q^i \wedge Q^j \wedge Q^k \otimes \gamma_{\mu\nu} \cdot Q_k \\
&\quad + 2\gamma_{ij}^\mu P^\mu \wedge P^\nu \wedge Q^i \wedge Q^j \otimes P_\nu \neq 0 ,
\end{aligned} \tag{58}$$

whence $H^3(I; \mathfrak{g})^s = 0$ and the infinitesimal deformation is unobstructed. In fact, since $H^0(I; \mathfrak{g})^s = 0$, we also see that $H^3(\mathfrak{g}; \mathfrak{g}) = 0$.

Integrating the infinitesimal deformation, we find the following one-parameter (t) family of Lie superalgebras containing the M2 Killing superalgebra:

$$\begin{aligned}
[Q_i, Q_j] &= \gamma_{ij}^\mu P_\mu - 2t\gamma_{ij}^{ab} L_{ab} \\
[P_\mu, P_\nu] &= 16t\varepsilon_{\mu\nu}^\rho P_\rho \\
[P_\mu, Q_i] &= -8t\gamma_\mu \cdot Q_i ,
\end{aligned} \tag{59}$$

¹There are other choices for representative cocycle, of course. There is a choice where the $P \wedge P \otimes P$ term is absent and one might be puzzled at the fact that this seems to imply that there is no deformation to the bosonic subalgebra; however upon integrating that infinitesimal deformation, one is ineluctably led to adding those terms.

where we have omitted the brackets involving the semisimple generators, since these do not deform. The parameter t is mostly fictitious: the resulting Lie algebras belong to three isomorphism classes corresponding to $t > 0$, $t = 0$ and $t < 0$. Indeed, let us rescale the generators $P_\mu \mapsto P'_\mu = \mu_P P_\mu$ and $Q_i \mapsto Q'_i = \mu_Q Q_i$, while keeping $L'_{\mu\nu} = L_{\mu\nu}$ and $L'_{ab} = L_{ab}$ fixed. Then choosing $\mu_P = -\frac{1}{8t}$ and $\mu_Q = \frac{1}{\sqrt{8|t|}}$ and dropping primes, we arrive at the following normalised form for the superalgebra (for $t \neq 0$):

$$\begin{aligned} [Q_i, Q_j] &= \pm \left(\gamma_{ij}^\mu P_\mu + \frac{1}{4} \gamma_{ij}^{ab} L_{ab} \right) \\ [P_\mu, P_\nu] &= -2\varepsilon_{\mu\nu}{}^\rho P_\rho \\ [P_\mu, Q_i] &= \gamma_\mu \cdot Q_i, \end{aligned} \tag{60}$$

where the sign is minus the sign of t . The superalgebras for $t < 0$ and $t > 0$ are different real forms of the same complex Lie superalgebra. In fact, given any real Lie superalgebra, multiplying the odd generators by i gives another real Lie superalgebra, reminiscent of the duality present in riemannian symmetric spaces. We notice that the Lie subalgebra spanned by $L_{\mu\nu}$ and P_μ is isomorphic to $\mathfrak{so}(2, 2)$. This is easy to see as follows. The P_μ span a simple ideal isomorphic to $\mathfrak{so}(2, 1)$ and the $L_{\mu\nu}$ span a Lie algebra also isomorphic to $\mathfrak{so}(2, 1)$. Therefore their joint span is a semidirect product of $\mathfrak{so}(2, 1)$ by $\mathfrak{so}(2, 1)$. However, simple Lie algebras admit no outer derivations, whence this semidirect product is actually isomorphic to a direct product, whence $L_{\mu\nu}$ and P_μ span a Lie subalgebra isomorphic to $\mathfrak{so}(2, 1) \oplus \mathfrak{so}(2, 1) \cong \mathfrak{so}(2, 2)$. This isomorphism can be made manifest by noticing that P_μ and $R_\mu := P_\mu - \varepsilon_\mu{}^{\nu\rho} L_{\nu\rho}$ are commuting $\mathfrak{so}(2, 1)$ -subalgebras. In particular, R_μ acts trivially on the supercharges. The Lie superalgebra spanned by P_μ , Q_α and L_{ab} is isomorphic to a real form of the classical Lie superalgebra $D(4, 1)$ in [39]. Hence abstractly as a Lie superalgebra the deformed M2-brane superalgebra is isomorphic to a real form of $A_1 \oplus D(4, 1)$. The $\mathfrak{so}(2, 2)$ subalgebra of the deformed superalgebra suggests that quantum corrections curve the brane worldvolume to AdS_3 with the quantum parameter being related to the curvature of the AdS_3 . Another possibility, currently being investigated [26], would be that this deformation is the Killing superalgebra of a one-parameter family of *classical* half-BPS M2-brane backgrounds where the M2-brane wraps an AdS_3 .

4.2. Rigidity of the M5-brane Killing superalgebra. The Killing superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of the M5-brane is the subalgebra of the Poincaré superalgebra defined as follows. First we split the 11-dimensional lorentzian vector space $V = W \oplus W^\perp$, where W is six-dimensional lorentzian. The subalgebra of the Lorentz algebra which preserves this split is $\mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp)$. Then $\mathfrak{g}_0 = \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp) \oplus W$. The odd part of the superalgebra is the subspace Δ of the $\mathfrak{so}(V)$ spinor module $\Delta(V)$ consisting of those spinors ψ for which $\nu_W \cdot \psi = \psi$. The volume element ν_W is skew-adjoint with respect to the invariant symplectic form and satisfies $\nu_W^2 = 1$. As an $\mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp)$ -module, this is $[\Delta(W)_+ \otimes \Delta(W^\perp)]$, where the chirality condition on the $\mathfrak{so}(W)$ spinors is precisely the supersymmetry projection condition $\nu_W \cdot \psi = \psi$, and where the brackets denote the underlying real representation of the product of quaternionic representations $\Delta(W)_+$ and $\Delta(W^\perp)$, each of which has four complex dimensions. The resulting superalgebra has dimension $(31|16)$.

As before, we let I be the supertranslation ideal isomorphic to $W \oplus \Delta$, so that again $\mathfrak{s} = \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp)$. We will let P_μ and Q_i denote a basis for I and $L_{\mu\nu}$ and L_{ab} be a basis for \mathfrak{s} . As before we will let P^μ and Q^i denote the bases for I^* canonically dual to P_μ and Q_i , respectively.

In this basis, the Lie brackets are formally the same as those in (44) after suitably reinterpreting the symbols.

There are no \mathfrak{s} -invariant elements in \mathfrak{g} , whence there are no invariant cochains. The space C^1 of invariant 1-cochains is two-dimensional, spanned by the identity maps $W \rightarrow W$ and $\Delta \rightarrow \Delta$. The representative cochains are

$$P^\mu \otimes P_\mu \quad \text{and} \quad Q^i \otimes Q_i . \quad (61)$$

The space C^2 of invariant 2-cochains is three-dimensional, spanned by the identity map $\Lambda^2 W \rightarrow \Lambda^2 W$, the Clifford product $W \otimes \Delta \rightarrow \Delta$, and the squaring map $S^2 \Delta \rightarrow W$, with representative cochains

$$P^\mu \wedge P^\nu \otimes L_{\mu\nu} , \quad P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i \quad \text{and} \quad \gamma_{ij}^\mu Q^i \wedge Q^j \otimes P_\mu . \quad (62)$$

The squaring map $S^2 \Delta \rightarrow \Lambda^2 W \oplus \Lambda^2 W^\perp$ is zero because of the projection condition on the spinors.

The calculation of the differential on C^1 and C^2 is very similar to those for the M2 brane and can almost be read off from those. There is a 1-cocycle

$$2P^\mu \otimes P_\mu - Q^i \otimes Q_i , \quad (63)$$

whence $H^1(\mathfrak{g}; \mathfrak{g}) \cong \mathbb{R}$ in the absence of any coboundaries. As before, this outer derivation can be interpreted as dilatations with the same weights as in the case of the Minkowski and the M2-brane Killing superalgebras.

We learn that $\gamma_{ij}^\mu Q^i \wedge Q^j \otimes P_\mu$ is the only 2-coboundary, whereas the calculations

$$d(P^\mu \wedge P^\nu \otimes L_{\mu\nu}) = \gamma_{ij}^\mu Q^i \wedge Q^j \wedge P^\nu \otimes L_{\mu\nu} + \frac{1}{2} P^\mu \wedge P^\nu \wedge Q^i \otimes \gamma_{\mu\nu} \cdot Q_i \quad (64)$$

and

$$d(P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i) = \frac{1}{2} \gamma_{ij}^\mu Q^i \wedge Q^j \wedge Q^k \otimes \gamma_\mu \cdot Q_k \quad (65)$$

show that there are no further 2-cocycles. Therefore $H^2(\mathfrak{g}; \mathfrak{g}) = 0$ and the M5 brane Killing superalgebra is rigid.

5. SUPERALGEBRA DEFORMATIONS OF PURELY GRAVITATIONAL BACKGROUNDS

In this section we tackle the Lie superalgebra deformations of the purely gravitational $\frac{1}{2}$ -BPS backgrounds: the Kaluza–Klein monopole [42, 43, 44] and the M-wave [45]. In the absence of flux, the Killing spinors are parallel in these backgrounds. This means that the Lie bracket of supercharges consists of parallel vectors and hence of translations.

5.1. A deformation of the M-wave Killing superalgebra. The Killing superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of the maximally symmetric $\frac{1}{2}$ -BPS M-wave is the (38|16)-dimensional subsuperalgebra of the Poincaré superalgebra defined as follows. We first split the eleven-dimensional lorentzian vector space $V = W \oplus W^\perp$, where W is a two-dimensional lorentzian subspace and W^\perp is the perpendicular euclidean space, which can be interpreted as the transverse space to the wave front. We can write $W = W_+ \oplus W_-$, where W_\pm are isotropic one-dimensional subspaces, with W_+ spanned by the parallel vector. The even subalgebra $\mathfrak{g}_0 = \mathfrak{so}(W^\perp) \oplus W$ and the odd subspace $\mathfrak{g}_1 = \Delta$, with Δ the sixteen-dimensional subspace of the space of eleven-dimensional spinors defined as the kernel of Clifford multiplication by W_+ . As before, we take $I \cong W \oplus \Delta$ to be the supertranslation ideal and $\mathfrak{s} = \mathfrak{so}(W^\perp)$ to be the semisimple factor. Let e_\pm span W_\pm , e_a span W and ε_i span Δ . The corresponding basis for I is given by P_\pm and Q_i , with P^\pm and Q^i denoting the canonical dual basis for I^* . We will let L_{ab} span $\mathfrak{so}(W^\perp)$.

In this basis, the Lie brackets take the form

$$[L_{ab}, Q_i] = \frac{1}{2}\gamma_{ab} \cdot Q_i \quad \text{and} \quad [Q_i, Q_j] = \Omega_{ij}P_+, \quad (66)$$

in addition to the ones of \mathfrak{s} , where the bilinear form $\Omega_{ij} := \langle \varepsilon_i, \gamma^+ \cdot \varepsilon_j \rangle = \langle \varepsilon_i, \gamma_- \cdot \varepsilon_j \rangle$ is symmetric and positive-definite on Δ . As representations of $\mathfrak{so}(W^\perp)$, we have

$$S^2\Delta \cong \mathbb{R} \oplus W^\perp \oplus \Lambda^4 W^\perp. \quad (67)$$

The differential of the invariant subcomplex (C^\bullet, d) is defined by its action on the elements of I^* and of \mathfrak{g} as an I -module; that is,

$$\begin{aligned} dP^- &= 0 & dP^+ &= \frac{1}{2}\Omega_{ij}Q^i \wedge Q^j \\ dQ^i &= 0 & dQ_i &= -\Omega_{ij}Q^j \otimes P_+ \\ dP_\pm &= 0 & dL_{ab} &= \frac{1}{2}Q^i \otimes \gamma_{ab} \cdot Q_i. \end{aligned} \quad (68)$$

The space of invariant 0-cochains is two-dimensional, spanned by P_\pm . They are central elements in \mathfrak{g} , whence cocycles. Since there are no coboundaries, $\dim H^0(\mathfrak{g}; \mathfrak{g}) = 2$ and $\dim B^1 = 0$.

The space of invariant 1-cochains is 5-dimensional, consisting of the 4-dimensional subspace $\text{End}(W)$ and the one-dimensional subspace spanned by the identity map $\Delta \rightarrow \Delta$. The corresponding cochains are $Q^i \otimes Q_i$ and $P^\pm \otimes P_\pm$ with uncorrelated signs. The differential $d : C^1 \rightarrow C^2$ is such that $P^- \otimes P_\pm$ are cocycles and

$$d(P^+ \otimes P_\pm) = \frac{1}{2}\Omega_{ij}Q^i \wedge Q^j \otimes P_\pm, \quad (69)$$

and

$$d(Q^i \otimes Q_i) = \Omega_{ij}Q^i \wedge Q^j \otimes P_+. \quad (70)$$

Therefore we see that $\dim Z^1 = 3$, whence $\dim B^2 = 2$, with basis $P^- \otimes P_\pm$ and

$$Q^i \otimes Q_i + 2P^+ \otimes P_+. \quad (71)$$

As there are no coboundaries, we see that $\dim H^1(\mathfrak{g}; \mathfrak{g}) = 3$.

The space of invariant 2-cochains is six-dimensional with basis

$$P^+ \wedge P^- \otimes P_\pm \quad P^\pm \wedge Q^i \otimes Q_i \quad \Omega_{ij}Q^i \wedge Q^j \otimes P_\pm. \quad (72)$$

The differential $d : C^2 \rightarrow C^3$ is given by

$$\begin{aligned} d(P^+ \wedge P^- \otimes P_\pm) &= \frac{1}{2}\Omega_{ij}P^- \wedge Q^i \wedge Q^j \otimes P_\pm \\ d(P^- \wedge Q^i \otimes Q_i) &= -\Omega_{ij}P^- \wedge Q^i \wedge Q^j \otimes P_+ \\ d(P^+ \wedge Q^i \otimes Q_i) &= -\Omega_{ij}P^+ \wedge Q^i \wedge Q^j \otimes P_+ + \frac{1}{2}\Omega_{ij}Q^i \wedge Q^j \wedge Q^k \otimes Q_k, \end{aligned} \quad (73)$$

and by $d(\Omega_{ij}Q^i \wedge Q^j \otimes P_\pm) = 0$. These two cocycles are also coboundaries and the only other cocycle is

$$P^- \wedge Q^i \otimes Q_i + 2P^+ \wedge P^- \otimes P_+. \quad (74)$$

In other words, $\dim Z^2 = 3$ and, since $\dim B^2 = 2$, we see that $\dim H^2(\mathfrak{g}; \mathfrak{g}) = 1$ with the above representative cocycle. This means that there is a one-dimensional space of infinitesimal deformations. It is easy to show by an explicit computation that this infinitesimal deformation is unobstructed and we end up with the following one-parameter (t) family of Lie superalgebras containing the M-wave Killing superalgebra:

$$[Q_i, Q_j] = \Omega_{ij}P_+ \quad [P_-, Q_i] = -tQ_i \quad [P_+, P_-] = 2tP_+, \quad (75)$$

where we have omitted the brackets involving \mathfrak{s} , as these remain undeformed. By rescaling P_- we see that all superalgebras for $t \neq 0$ are isomorphic, whence we can let t above take only two values: 0 and 1. In the former case, it is the original M-wave Killing superalgebra, whereas in the latter case it is a deformation

$$\begin{aligned} [Q_i, Q_j] &= \Omega_{ij} P_+ \\ [P_-, Q_i] &= Q_i \\ [P_-, P_+] &= 2P_+ , \end{aligned} \tag{76}$$

perhaps induced by quantum corrections or perhaps belonging to a one-parameter family of backgrounds which tends to the M-wave under some geometric limit contracting its Killing superalgebra.

5.2. Deformations of the Kaluza–Klein monopole Killing superalgebra. The Killing superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of the $\frac{1}{2}$ -BPS Kaluza–Klein monopole is the (32|16)-dimensional sub-superalgebra of the Poincaré superalgebra defined as follows. Let us split the 11-dimensional lorentzian vector space $V = W \oplus W^\perp$, where W is a 7-dimensional lorentzian subspace and W^\perp is the perpendicular 4-dimensional euclidean space. The even subalgebra $\mathfrak{g}_0 = \mathfrak{so}(W) \oplus W \oplus \mathfrak{u}(W^\perp)$, where $\mathfrak{u}(W^\perp) \subset \mathfrak{so}(W^\perp)$ is the 4-dimensional subalgebra preserving a self-dual hermitian structure on W^\perp . If the hermitian structure is defined by the metric and a compatible complex structure J , then the associated 2-form ω on W^\perp is anti-self dual: $\omega \in \Lambda_-^2 W^\perp$. Then the subalgebra $\mathfrak{u}(W^\perp)$ is spanned by the self-dual two forms together with ω . Under this decomposition, we will write $\mathfrak{u}(W^\perp) = \mathfrak{su}(W^\perp) \oplus \mathbb{R}\omega$.

The odd subspace is $\mathfrak{g}_1 \cong \Delta$, with Δ the sixteen-dimensional subspace of the space of eleven-dimensional spinors defined by the projection condition $\nu_{W^\perp} \cdot \psi = -\psi$. We take $I \cong W \oplus \Delta \oplus \mathbb{R}\omega$ to be the ideal and $\mathfrak{s} = \mathfrak{so}(W) \oplus \mathfrak{su}(W^\perp)$ to be the semisimple factor. We will let P_μ, Q_i and ω span I and $L_{\mu\nu} := L(e_\mu \wedge e_\nu)$, for $\mu < \nu$, and $L_{ab}^+ := L(e_a \wedge e_b + \star(e_a \wedge e_b))$, for $a < b$, span \mathfrak{s} . We let P^μ, Q^i and ω^* denote the canonical dual basis for I^* .

In this basis, the Lie brackets are given by

$$\begin{aligned} [L_{\mu\nu}, Q_i] &= \frac{1}{2} \gamma_{\mu\nu} \cdot Q_i & [L_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \\ [L_{ab}^+, Q_i] &= \frac{1}{2} \gamma_{ab}^+ \cdot Q_i & [Q_i, Q_j] &= \gamma_{ij}^\mu P_\mu \end{aligned} \tag{77}$$

in addition to the ones of \mathfrak{s} .

As an \mathfrak{s} -module, $\Delta = [\Delta^{1,6} \otimes \Delta_-^4]$, with $\Delta^{1,6}$ the complex 8-dimensional quaternionic spinorial representation of $\text{Spin}(1,6)$, Δ_-^4 the complex 2-dimensional quaternionic representation of $\text{Spin}(4)$ consisting of negative chirality spinors, and where as usual the brackets indicate the underlying real subrepresentation. As an \mathfrak{s} -module,

$$S^2 \Delta \cong W \oplus \Lambda^2 W \oplus \Lambda_+^2 W^\perp \oplus \left(\Lambda^3 W \otimes \Lambda_+^2 W^\perp \right) . \tag{78}$$

It is now possible to list the \mathfrak{s} -invariant cochains. The centre of \mathfrak{g} is spanned by ω , whence $\dim C^0 = \dim Z^0 = 1$ and in the absence of coboundaries $\dim H^0(\mathfrak{g}; \mathfrak{g}) = 1$. This also shows that $\dim B^1 = 0$. The invariant 1-cochains are induced by the identity maps $W \rightarrow W$, $\Delta \rightarrow \Delta$ and $\mathbb{R}\omega \rightarrow \mathbb{R}\omega$, yielding the following cochains

$$P^\mu \otimes P_\mu \quad Q^i \otimes Q_i \quad \omega^* \otimes \omega , \tag{79}$$

whence $\dim C^1 = 3$.

The invariant 2-cochains are given by the natural isomorphism $\Lambda^2 W \rightarrow \mathfrak{so}(W)$, Clifford multiplication $W \otimes \Delta \rightarrow \Delta$, the squaring maps $S^2 \Delta \rightarrow W \oplus \mathfrak{so}(W) \oplus \mathfrak{su}(W^\perp)$, as well as the

isomorphisms $\mathbb{R}\omega \otimes W \rightarrow W$ and $\mathbb{R}\omega \otimes \Delta \rightarrow \Delta$ induced by the identity maps on W and Δ . The corresponding cochains are

$$\begin{aligned} \gamma_{ij}^\mu Q^i \wedge Q^j \otimes P_\mu & & P^\mu \wedge P^\nu \otimes L_{\mu\nu} \\ \gamma_{ij}^{\mu\nu} Q^i \wedge Q^j \otimes L_{\mu\nu} & & P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i \\ \left(\gamma_+^{ab}\right)_{ij} Q^i \wedge Q^j \otimes L_{ab}^+ & & \omega^* \wedge P^\mu \otimes P_\mu \\ & & \omega^* \wedge Q^i \otimes Q_i, \end{aligned} \quad (80)$$

whence $\dim C^2 = 7$.

The space of invariant 3-cochains is 9-dimensional, spanned by the following cochains associated to the natural maps:

$$\begin{aligned} \omega^* \wedge P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i & & P^\mu \wedge P^\nu \wedge Q^i \otimes \gamma_{\mu\nu} \cdot Q_i \\ \omega^* \wedge P^\mu \wedge P^\nu \otimes L_{\mu\nu} & & \gamma_{ij}^\mu P^\nu \wedge Q^i \wedge Q^j \otimes L_{\mu\nu} \\ \gamma_{ij}^\mu \omega^* \wedge Q^i \wedge Q^j \otimes P_\mu & & (\gamma_\mu^\nu)_{ij} P^\mu \wedge Q^i \wedge Q^j \otimes P_\nu \\ \gamma_{ij}^{\mu\nu} \omega^* \wedge Q^i \wedge Q^j \otimes L_{\mu\nu} & & \gamma_{ij}^\mu Q^i \wedge Q^j \wedge Q^k \otimes \gamma_\mu \cdot Q_k, \\ \left(\gamma_+^{ab}\right)_{ij} \omega^* \wedge Q^i \wedge Q^j \otimes L_{ab}^+ & & \end{aligned} \quad (81)$$

where the absence of the cochain

$$\left(\gamma_+^{ab}\right)_{ij} Q^i \wedge Q^j \wedge Q^k \otimes \gamma_{ab}^+ \cdot Q_k \quad (82)$$

is explained by the fact that it is in the span of the above cochains by virtue of the Fierz identity (43), and the absence of the cochain

$$\gamma_{ij}^{\mu\nu} Q^i \wedge Q^j \wedge Q^k \otimes \gamma_{\mu\nu} \cdot Q_k \quad (83)$$

is explained by the Fierz identity

$$\frac{1}{2} \langle \psi, \gamma^{\mu\nu} \psi \rangle \gamma_{\mu\nu} \psi = -3 \langle \psi, \gamma^\mu \psi \rangle \gamma_\mu \psi, \quad (84)$$

for $\psi \in \Delta$. Together with the Fierz identity (43), we also obtain

$$\frac{1}{2} \langle \psi, \gamma_+^{ab} \psi \rangle \gamma_{ab}^+ \psi = -2 \langle \psi, \gamma^\mu \psi \rangle \gamma_\mu \psi. \quad (85)$$

The differential of the invariant subcomplex (C^\bullet, d) is defined by its action on the elements of I^* and of \mathfrak{g} as an I -module; that is,

$$\begin{aligned} dP_\mu &= 0 & dP^\mu &= \frac{1}{2} \gamma_{ij}^\mu Q^i \wedge Q^j \\ dQ^i &= 0 & dQ_i &= -\gamma_{ij}^\mu Q^j \otimes P_\mu \\ d\omega^* &= 0 & dL_{\mu\nu} &= \eta_{\mu\rho} P^\rho \otimes P_\nu - \eta_{\nu\rho} P^\rho \otimes P_\mu + \frac{1}{2} Q^i \otimes \gamma_{\mu\nu} \cdot Q_i \\ d\omega &= 0 & dL_{ab}^+ &= \frac{1}{2} Q^i \otimes \gamma_{ab}^+ \cdot Q_i. \end{aligned} \quad (86)$$

The unique invariant 0-cochain ω is a cocycle, whence $\dim H^0(\mathfrak{g}; \mathfrak{g}) = 1$. The differential $d : C^1 \rightarrow C^2$ is given by

$$\begin{aligned} d(P^\mu \otimes P_\mu) &= -\frac{1}{2} \gamma_{ij}^\mu Q^i \wedge Q^j \otimes P_\mu \\ d(Q^i \otimes Q_i) &= \gamma_{ij}^\mu Q^i \wedge Q^j \otimes P_\mu \\ d(\omega^* \otimes \omega) &= 0. \end{aligned} \quad (87)$$

The space of cocycles is 2-dimensional, spanned by $\omega^* \otimes \omega$ and

$$2P^\mu \otimes P_\mu - Q^i \otimes Q_i . \quad (88)$$

Since there are no coboundaries, $\dim H^1(\mathfrak{g}; \mathfrak{g}) = 2$. This calculation also shows that $\dim B^2 = 1$, spanned by $\gamma_{ij}^\mu Q^i \wedge Q^j \otimes P_\mu$.

The differential $d : C^2 \rightarrow C^3$ on the remaining cochains is given by

$$\begin{aligned} d(P^\mu \wedge P^\nu \otimes L_{\mu\nu}) &= \gamma_{ij}^\mu Q^i \wedge Q^j \wedge P^\nu \otimes L_{\mu\nu} + \frac{1}{2} P^\mu \wedge P^\nu \wedge Q^i \otimes \gamma_{\mu\nu} \cdot Q_i \\ d(\omega^* \wedge P^\mu \otimes P_\mu) &= -\frac{1}{2} \gamma_{ij}^\mu \omega^* \wedge Q^i \wedge Q^j \otimes P_\mu \\ d(\omega^* \wedge Q^i \otimes Q_i) &= -\gamma_{ij}^\mu \omega^* \wedge Q^i \wedge Q^j \otimes P_\mu \\ d(P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i) &= (\gamma_\mu^\nu)_{ij} P^\mu \wedge Q^i \wedge Q^j \otimes P_\nu + \frac{1}{2} \gamma_{ij}^\mu Q^i \wedge Q^j \wedge Q^k \otimes \gamma_\mu \cdot Q_k \\ d(\gamma_{ij}^{\mu\nu} Q^i \wedge Q^j \otimes L_{\mu\nu}) &= 2(\gamma_\mu^\nu)_{ij} P^\mu \wedge Q^i \wedge Q^j \otimes P_\nu - 3\gamma_{ij}^\mu Q^i \wedge Q^j \wedge Q^k \otimes \gamma_\mu \cdot Q_k \\ d\left(\left(\gamma_+^{ab}\right)_{ij} Q^i \wedge Q^j \otimes L_{ab}^+\right) &= -2\gamma_{ij}^\mu Q^i \wedge Q^j \wedge Q^k \otimes \gamma_\mu \cdot Q_k , \end{aligned} \quad (89)$$

where we have used equations (84) and (85). It is not hard to show that there are two linearly independent cocycles:

$$2\omega^* \wedge P^\mu \otimes P_\mu - \omega^* \wedge Q^i \otimes Q_i , \quad (90)$$

and

$$P^\mu \wedge Q^i \otimes \gamma_\mu \cdot Q_i - \frac{1}{2} \gamma_{ij}^{\mu\nu} Q^i \wedge Q^j \otimes L_{\mu\nu} + \left(\gamma_+^{ab}\right)_{ij} Q^i \wedge Q^j \otimes L_{ab}^+ , \quad (91)$$

whence $\dim H^2(\mathfrak{g}; \mathfrak{g}) = 2$. This gives rise to a two-dimensional space of infinitesimal deformations of the Killing superalgebra.

The deformation corresponding to the cocycle (90) is unobstructed, which gives rise to a one-parameter (t) deformation of the Killing superalgebra of the Kaluza–Klein monopole, given by

$$[Q_i, Q_j] = \gamma_{ij}^\mu P_\mu \quad [\omega, Q_i] = tQ_i \quad [\omega, P_\mu] = 2tP_\mu , \quad (92)$$

together with the brackets involving \mathfrak{s} , which do not deform. By rescaling ω , we see that there are two isomorphism classes of Lie superalgebras in this family, corresponding to the values $t = 0$, which is the original Killing superalgebra, and $t = 1$, given by

$$\begin{aligned} [Q_i, Q_j] &= \gamma_{ij}^\mu P_\mu \\ [\omega, Q_i] &= Q_i \\ [\omega, P_\mu] &= 2P_\mu , \end{aligned} \quad (93)$$

in addition to the brackets involving \mathfrak{s} . In essence ω acts now as homotheties on the 7-dimensional Minkowski spacetime as well as rotations in Taub-NUT. Curiously it is now seen to generate a subgroup \mathbb{R} and not a compact subgroup $U(1)$. The form of the superalgebra would suggest a geometry which is no longer a metric product but rather a *warped* product where the size of the Minkowski factor now depends on the angular variable in the Taub-NUT; however this cannot be the case because the identifications in the angular variable. This is reminiscent of a non-geometric background [46] and might be related to the discussion in [47, 48].

The deformation corresponding to the cocycle (91) is also unobstructed, but unlike the previous case, this requires adding a term of order t^2 to the $[P, P]$ bracket. The one-parameter

deformation is given by

$$\begin{aligned} [Q_i, Q_j] &= \gamma_{ij}^\mu P_\mu + t\gamma_{ij}^{\mu\nu} L_{\mu\nu} - 2t \left(\gamma_+^{ab} \right)_{ij} L_{ab}^+ \\ [P_\mu, Q_i] &= -t\gamma_\mu \cdot Q_i \\ [P_\mu, P_\nu] &= 4t^2 L_{\mu\nu} . \end{aligned} \tag{94}$$

For $t \neq 0$, we may rescaling P_μ by $\frac{1}{2t}$ and Q_i by $\frac{1}{\sqrt{2|t|}}$ in order to bring the Lie algebra to the following form

$$\begin{aligned} [Q_i, Q_j] &= \pm \left(\gamma_{ij}^\mu P_\mu + \frac{1}{2}\gamma_{ij}^{\mu\nu} L_{\mu\nu} - \left(\gamma_+^{ab} \right)_{ij} L_{ab}^+ \right) \\ [P_\mu, Q_i] &= -\frac{1}{2}\gamma_\mu \cdot Q_i \\ [P_\mu, P_\nu] &= L_{\mu\nu} , \end{aligned} \tag{95}$$

where the sign is the sign of t . These Lie superalgebras are real forms of the classical Lie superalgebra $D(4, 1)$ augmented by the central element ω . In particular the even subalgebra is $\mathfrak{so}(2, 6) \oplus \mathfrak{u}(2)$, which suggests that the Minkowski factor deforms to AdS_7 .

Any other linear combination of the cocycles (90) and (91) is obstructed. This is easy to see because the weights of Q and P relative to the adjoint action of ω implied by the cocycle (90) is incompatible with the $[P, Q]$ bracket implied by the cocycle (91).

6. CONCLUSION

In this paper we have started the study of the deformations of Killing superalgebras of supersymmetric eleven-dimensional backgrounds. Our motivation is that the deformed Killing superalgebra gives us hints about the possible quantum corrections a classical background might undergo or, keeping within the classical theory, about possible backgrounds which can tend to the original one under a geometric limit—the rationale being that the Killing superalgebra contracts under such a limit whence it must be found among the deformations of the contracted algebra.

We have studied backgrounds with either maximal supersymmetry or half-BPS. We have shown the rigidity of the Killing superalgebras of the maximally supersymmetric backgrounds we have studied, namely Minkowski spacetime and the Freund–Rubin backgrounds. This agrees with the heuristic idea that supersymmetry tends to rigidify the geometry and with known results about the fact that these backgrounds do not admit quantum corrections [41]. We have not attempted to classify the deformations of the Killing superalgebra of the Kowalski-Glikman background, since the semisimple factor is not large enough to allow a painless calculation, but since it is known that its superalgebra gets deformed [22, 23, 24, 25], we know that it will have at least two deformations: corresponding to the Killing superalgebras of the Freund–Rubin backgrounds. However I would hazard the “conjecture” that no further deformations exist.

Among the half-BPS backgrounds considered in this paper, the Killing superalgebra of the M5-brane is rigid, whereas that of the M2-brane, the M-wave and the Kaluza–Klein monopole admits nontrivial deformations given up to isomorphism, by the Lie superalgebras in equations (60), (76), (93) and (95). In particular, the structure of the Lie superalgebras in (60) and (95) suggest that the worldvolume of the M2-brane acquires constant negative curvature and so does that of the Minkowski factor in the Kaluza–Klein monopole. In a forthcoming paper [49], in which we study the Killing superalgebra deformations of some ten-dimensional

supergravity backgrounds, we present evidence that deformations behave well under Kaluza–Klein reduction, suggesting a geometric interpretation for the deformed superalgebras found here and also in that paper. We are currently investigating whether the existence of the deformations found in this paper can be explained within the context of supergravity [26].

ACKNOWLEDGMENTS

I have benefited from discussions with Chris Hull, Hyakutake Yoshifumi, Patricia Ritter, Joan Simón and Bert Vercnocke.

REFERENCES

- [1] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull, and B. Spence, “Branes at conical singularities and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 1249–1286, [hep-th/9808014](#).
- [2] J. P. Gauntlett, R. Myers, and P. K. Townsend, “Supersymmetry of rotating branes,” *Phys. Rev.* **D59** (1999) 025001, [hep-th/9809065](#).
- [3] J. P. Gauntlett, R. Myers, and P. K. Townsend, “Black holes of $D = 5$ supergravity,” *Class. Quant. Grav.* **16** (1999) 1–21, [hep-th/9810204](#).
- [4] P. K. Townsend, “Killing spinors, supersymmetries and rotating intersecting branes,” in *Novelties in string theory (Göteborg, 1998)*, pp. 177–182. World Sci. Publishing, River Edge, NJ, 1999. [hep-th/9901102](#).
- [5] J. M. Figueroa-O'Farrill, “On the supersymmetries of Anti-de Sitter vacua,” *Class. Quant. Grav.* **16** (1999) 2043–2055, [hep-th/9902066](#).
- [6] J. M. Figueroa-O'Farrill and G. Papadopoulos, “Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory,” *J. High Energy Phys.* **06** (2001) 036, [hep-th/0105308](#).
- [7] M. Blau, J. M. Figueroa-O'Farrill, C. M. Hull, and G. Papadopoulos, “A new maximally supersymmetric background of type IIB superstring theory,” *J. High Energy Phys.* **01** (2002) 047, [hep-th/0110242](#).
- [8] N. Alonso-Alberca, E. Lozano-Tellechea, and T. Ortín, “Geometric construction of Killing spinors and supersymmetry algebras in homogeneous spacetimes,” *Class. Quant. Grav.* **19** (2002) 6009–6024, [hep-th/0208158](#).
- [9] J. M. Figueroa-O'Farrill, P. Meessen, and S. Philip, “Supersymmetry and homogeneity of M-theory backgrounds,” *Class. Quant. Grav.* **22** (2005) 207–226, [hep-th/0409170](#).
- [10] J. M. Figueroa-O'Farrill, E. Hackett-Jones, and G. Moutsopoulos, “The Killing superalgebra of ten-dimensional supergravity backgrounds,” *Class. Quant. Grav.* **24** (2007) 3291–3308, [hep-th/0703192](#).
- [11] J. Gillard, G. Papadopoulos, and D. Tsimplis, “Anomaly, fluxes and $(2, 0)$ heterotic-string compactifications,” *J. High Energy Phys.* (2003), no. 6, 035, 25 pp. (electronic), [hep-th/0304126](#).
- [12] H. Lu, C. N. Pope, and K. S. Stelle, “Higher-order corrections to non-compact Calabi-Yau manifolds in string theory,” *JHEP* **07** (2004) 072, [hep-th/0311018](#).
- [13] H. Lu, C. N. Pope, K. S. Stelle, and P. K. Townsend, “Supersymmetric deformations of G_2 manifolds from higher-order corrections to string and M-theory,” *JHEP* **10** (2004) 019, [hep-th/0312002](#).
- [14] H. Lu, C. N. Pope, K. S. Stelle, and P. K. Townsend, “String and M-theory deformations of manifolds with special holonomy,” *JHEP* **07** (2005) 075, [hep-th/0410176](#).
- [15] Y. Hyakutake and S. Ogushi, “ R^4 corrections to eleven dimensional supergravity via supersymmetry,” *Phys. Rev.* **D74** (2006) 025022, [hep-th/0508204](#).
- [16] H. Lu, C. N. Pope, and K. S. Stelle, “Generalised holonomy for higher-order corrections to supersymmetric backgrounds in string and M-theory,” *Nucl. Phys.* **B741** (2006) 17–33, [hep-th/0509057](#).
- [17] Y. Hyakutake and S. Ogushi, “Higher derivative corrections to eleven dimensional supergravity via local supersymmetry,” *J. High Energy Phys.* **02** (2006) 068, [hep-th/0601092](#).
- [18] Y. Hyakutake, “Toward the determination of $R^3 F^2$ terms in M-theory,” [hep-th/0703154](#).
- [19] M. Gerstenhaber, “On the deformation of rings and algebras,” *Ann. of Math. (2)* **79** (1964) 59–103.
- [20] R. Penrose, “Any space-time has a plane wave as a limit,” in *Differential geometry and relativity*, pp. 271–275. Reidel, Dordrecht, 1976.
- [21] R. Güven, “Plane wave limits and T-duality,” *Phys. Lett.* **B482** (2000) 255–263, [hep-th/0005061](#).
- [22] M. Blau, J. M. Figueroa-O'Farrill, C. M. Hull, and G. Papadopoulos, “Penrose limits and maximal supersymmetry,” *Class. Quant. Grav.* **19** (2002) L87–L95, [hep-th/0201081](#).

- [23] M. Blau, J. M. Figueroa-O’Farrill, and G. Papadopoulos, “Penrose limits, supergravity and brane dynamics,” *Class. Quant. Grav.* **19** (2002) 4753–4805, [hep-th/0202111](#).
- [24] M. Hatsuda, K. Kamimura, and M. Sakaguchi, “Super-pp-wave algebra from super-AdS x S algebras in eleven-dimensions,” *Nucl. Phys.* **B637** (2002) 168–176, [hep-th/0204002](#).
- [25] S. Stanciu and J. M. Figueroa-O’Farrill, “Penrose limits of Lie branes and a Nappi–Witten braneworld,” *J. High Energy Phys.* **06** (2003) 025, [hep-th/0303212](#).
- [26] J. M. Figueroa-O’Farrill and P. D. Ritter, “Deformations of supergravity backgrounds.” Work in progress.
- [27] A. Nijenhuis and R. W. Richardson, Jr., “Deformations of Lie algebra structures,” *J. Math. Mech.* **17** (1967) 89–105.
- [28] D. A. Leites, “Cohomology of Lie superalgebras,” *Functional Anal. Appl.* **9** (1975), no. 4, 340–341.
- [29] D. B. Fuks, *Cohomology of infinite-dimensional Lie algebras*. Contemporary Soviet Mathematics. Consultants Bureau, New York, 1986.
- [30] C. Chevalley and S. Eilenberg, “Cohomology theory of Lie groups and Lie algebras,” *Trans. Amer. Math. Soc.* **63** (1948) 85–124.
- [31] G. Hochschild and J.-P. Serre, “Cohomology of Lie algebras,” *Ann. of Math. (2)* **57** (1953) 591–603.
- [32] J. M. Figueroa-O’Farrill, “Deformations of the Galilean algebra,” *J. Math. Phys.* **30** (1989), no. 12, 2735–2739.
- [33] B. Binetgar, “Cohomology and deformations of Lie superalgebras,” *Lett. Math. Phys.* **12** (1986), no. 4, 301–308.
- [34] K. C. Tripathy and M. K. Patra, “Cohomology theory and deformations of \mathbb{Z}_2 -graded Lie algebras,” *J. Math. Phys.* **31** (1990), no. 12, 2822–2831.
- [35] B. Zumino, “Nonlinear realization of supersymmetry in de Sitter space,” *Nucl. Phys.* **B127** (1977) 189–201.
- [36] P. Freund and M. Rubin, “Dynamics of dimensional reduction,” *Phys. Lett.* **B97** (1980) 233–235.
- [37] K. Pilch, P. van Nieuwenhuizen, and P. K. Townsend, “Compactification of $d=11$ supergravity on S^4 (or $11 = 7 + 4$, too),” *Nucl. Phys.* **B242** (1984) 377.
- [38] J. Kowalski-Glikman, “Vacuum states in supersymmetric Kaluza–Klein theory,” *Phys. Lett.* **134B** (1984) 194–196.
- [39] V. G. Kac, “A sketch of Lie superalgebra theory,” *Comm. Math. Phys.* **53** (1977), no. 1, 31–64.
- [40] V. D. Ljahovskii, “Stability of semisimple superalgebras,” *Teoret. Mat. Fiz.* **38** (1979), no. 1, 115–120.
- [41] R. Kallosh and A. Rajaraman, “Vacua of M-theory and string theory,” *Phys. Rev.* **D58** (1998) 125003, [hep-th/9805041](#).
- [42] R. Sorkin, “Kaluza–Klein monopole,” *Phys. Rev. Lett.* **51** (1983) 87–90.
- [43] D. Gross and M. Perry, “Magnetic monopoles in Kaluza–Klein theories,” *Nucl. Phys.* **B226** (1983) 29.
- [44] S. Han and I. Koh, “ $N=4$ remaining supersymmetry in a Kaluza–Klein monopole background in $D=11$ supergravity theory,” *Phys. Rev.* **D31** (1985) 2503.
- [45] C. M. Hull, “Exact pp-wave solutions of eleven-dimensional supergravity,” *Phys. Lett.* **139B** (1984) 39–41.
- [46] A. Dabholkar and C. Hull, “Generalised T-duality and non-geometric backgrounds,” *JHEP* **05** (2006) 009, [hep-th/0512005](#).
- [47] D. Tong, “NS5-branes, T-duality and worldsheet instantons,” *JHEP* **07** (2002) 013, [hep-th/0204186](#).
- [48] R. Gregory, J. A. Harvey, and G. W. Moore, “Unwinding strings and T-duality of Kaluza–Klein and H -monopoles,” *Adv. Theor. Math. Phys.* **1** (1997) 283–297, [hep-th/9708086](#).
- [49] J. M. Figueroa-O’Farrill and B. Vercnocke, “Killing superalgebra deformations of ten-dimensional backgrounds.” In preparation.

MAXWELL INSTITUTE AND SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, UK
E-mail address: J.M.Figueroa@ed.ac.uk